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THE ROYAL COUPLE CONCEALS THEIR MUTUAL RELATIONSHIP: A NONCOALESCENT TOEPLITZ FLOW

BY

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ABSTRACT

There exists a regular Toeplitz sequence over a finite alphabet, such that its orbit-closure in the shift system is not topologically coalescent. The notion of a Toeplitz array is introduced.

Introduction

Toeplitz sequences and Toeplitz flows have been investigated from many points of view and by many authors (see [B-K1], [B-K2] and [D-K-L] for entropy and centralizers, [O], [W] and [D2] for invariant measures, [I-L], [I] and [D-L] for spectral properties). This note is fully devoted to presenting a solution to a single problem concerning Toeplitz flows, but related to many areas in ergodic theory. Namely, we will show that not all Toeplitz flows are topologically coalescent, i.e., we construct an example of a Toeplitz flow which admits a noninvertible topological endomorphism. Moreover, the generating Toeplitz sequence is obtained regular. No other example of a noncoalescent minimal zero-dimensional flow is known to the author.

The notion of coalescence originating as a measure theoretical term has been also adapted to topological dynamics. The result of this note illustrates that sometimes the topological version appears to be stronger than the measure

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theoretical one. Note that each regular Toeplitz flow is measure theoretically isomorphic to a group rotation, and hence it is measure theoretically coalescent.

For positive results on topological coalescence for certain classes of Toeplitz flows we refer the reader to Section 4 in [D-K-L], where recent as well as previously known facts are presented.

It may be worth announcing that the notion of a Toeplitz array introduced in this paper can be easily generalized to, say, "Toeplitz group action" over a general locally compact abelian group, but this direction of investigation will not be developed here.

Preliminaries

By a **flow** we mean a pair $(X, \mathbf{G})$, where X is a compact metric space and $\mathbf{G}$ is a group of homeomorphisms of X. Recall that a flow $(X, \mathbf{G})$ is said to be **minimal** provided X contains no proper nonempty $\mathbf{G}$-invariant closed subsets. By the **orbit-closure** of an element $x \in X$ we mean the set

$$\overline{O}(x) = \overline{\{g(x) : g \in \mathbf{G}\}}.$$

$(X, \mathbf{G})$ is minimal if and only if it is equal to every orbit-closure $\overline{O}(x)$, with $x \in X$.

Throughout the remainder of this work we will be assuming that either $\mathbf{G}$ is isomorphic to the set of all integers $\mathbf{Z}$, in which case the notion of (X, T) will be used instead of $(X, \mathbf{G})$ (T represents the generating element of $\mathbf{G}$), or $\mathbf{G}$ is isomorphic to $\mathbf{Z}^2$. The boldface capitals will be used to denote these groups or their subsets, in contrast to other sets (including groups), mostly playing the role of the phase space X. $\mathbf{N}$ will denote the set of positive integers. Lower case boldface is reserved for elements of $\mathbf{Z}^2$.

Let (X, T) and (Y, S) be some flows and let $\pi: X \to Y$ be an onto continuous mapping such that $\pi \circ T = S \circ \pi$. We then say that (Y, S) is a **factor** of (X, T). The mapping π is called a **homomorphism** between the flows.

A homomorphism from (X, T) into itself is called an **endomorphism**. An **automorphism** is an invertible endomorphism. A flow (X, T) is called **coalescent** if all its endomorphisms are automorphisms.

In this note we will investigate **subshifts**, i.e., the flows (X, S), with X being a compact subset of $\Sigma^{\mathbf{Z}}$, where Σ is a finite set called an **alphabet**, and S denoting the left shift transformation.

Each homomorphism π between subshifts over a finite alphabet Σ is determined by a **code**, i.e., a function $\Pi\colon \Sigma^{2r+1} \to \Sigma$ ($r \in \mathbf{N}$ will be called the **radius** of the code) in the following way:

$$\pi\omega(i) = \Pi(\omega[i-r, i+r])$$

($\omega \in \Sigma^{\mathbf{Z}}, i \in \mathbf{Z}$). Codes are frequently written in the following convention: instead of

$$\Pi([\sigma_1\ \sigma_2\ \dots\ \sigma_{2r+1}]) = \sigma$$

we write

$$\frac{\sigma_1\ \sigma_2\ \dots\ \sigma_{2r+1}}{\sigma}$$

($\sigma, \sigma_1, \sigma_2, \dots, \sigma_{2r+1} \in \Sigma$), which allows one to view the image of a sequence below it, so that each block of the length $2r+1$ codes to the symbol appearing underneath its central entry.

Definition: (See [J-K] for the original equivalent formulation.) A sequence ω over a finite alphabet Σ is called **Toeplitz** if there exists a sequence of positive integers p_n (called **periods**) with the subgroups $p_n\mathbf{Z} \subset \mathbf{Z}$ decreasing, and a sequence of subsets $\mathbf{H}_n \subset \mathbf{Z}$ decreasing to the empty intersection, such that each $\mathbf{H}_n$ is p_n-periodic and ω restricted to $\mathbf{Z} \setminus \mathbf{H}_n$ is p_n-periodic.

The elements of $\mathbf{H}_n$ are called **n-holes**.

By a Toeplitz flow we mean a subshift $(\overline{O}(\omega), S)$ where ω is a Toeplitz sequence. It is known that every Toeplitz flow is minimal. The set $\overline{O}(\omega)$ consists of both Toeplitz sequences and nontoeplitz sequences for which a condition like in the definition holds except that the intersection of the sets $\mathbf{H}_n$ is nonempty (this intersection is then called the **aperiodic** part and its elements are the **holes of infinite level**; see [W] for the general reference on Toeplitz flows).

A Toeplitz sequence is called **regular** if the densities of $\mathbf{H}_n$ tend to zero. Regular Toeplitz sequences are important for their strong measure theoretical properties.

If a sequence p_n as above is given then the **p-adding machine** is the compact monothetic group G_p consisting of elements of the form

$$h = (h_n) \in \prod_{n=1}^{\infty} \{0, 1, \dots, p_n - 1\}$$

such that $h_{n+1} = h_n (\mathrm{mod}\, p_n)$. By identifying every such $h \in G_p$ which is constantly j starting at some n_0, with the positive integer j, and every such h for which $h_n = p_n - j$ starting at some n_0, with the negative integer $-j$, the group G_p can be viewed as a compactification of the integers. In this setting $1 = (1, 1, 1, \dots)$ is the generator of G_p.

It is known that the flow $(G_p, 1)$ (the adding machine with the translation by the generator) is a factor of each Toeplitz flow with the periods p_n (compare with the notion of the maximal uniformly continuous factor, see [W]; see also [D1] for the relation between G_p and the maximal uniformly continuous factor). It is worth noticing that the corresponding homomorphism π_0 from $\overline{O}(\omega)$ to G_p is 1-1 exactly on the subset of $\overline{O}(\omega)$ consisting of Toeplitz sequences.

An endomorphism π of a Toeplitz flow $(\overline{O}(\omega), S)$ induces an endomorphism of the adjacent adding machine: there exists translation by an element $h \in G_p$ of $(G_p, 1)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\overline{O}(\omega), S) & \xrightarrow{\pi} & (\overline{O}(\omega), S) \\ {\scriptstyle \pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \pi_0} \\ (G_p, 1) & \xrightarrow[h]{} & (G_p, 1). \end{array}$$

More precisely, the relation between $h = (h_n)$ and π is such that the set of n-holes in $\pi(\omega)$ equals $\mathbf{H}_n - h_n$.

A noninvertible endomorphism of a Toeplitz flow is said to be of the **first type** if the preimage of every Toeplitz sequence is a one-element set (then this element is also a Toeplitz sequence). Otherwise we have an endomorphism of the **second type**. The first type is "closer" to invertibility.

Toeplitz arrays

Let $\mathbf{G} = \mathbf{G}_{(\mathbf{p},\mathbf{q})}$ denote the subgroup of $\mathbf{Z}^2$ generated by two independent vectors $\mathbf{p}, \mathbf{q} \in \mathbf{Z}^2$. We say that a subset $\mathbf{K}$ of $\mathbf{Z}^2$ is $\mathbf{G}$-periodic if

$$\mathbf{K} = \mathbf{K} + \mathbf{v},$$

for all $\mathbf{v} \in \mathbf{G}$. Periodicity of arrays defined on $\mathbf{Z}^2$ will be understood accordingly.

Definition: An array $\Omega(\mathbf{u})$, $\mathbf{u} \in \mathbf{Z}^2$ over a finite alphabet Σ is called **Toeplitz** if there exists a decreasing sequence of subgroups $\mathbf{G}_n = \mathbf{G}_{(\mathbf{p}_n,\mathbf{q}_n)}$, and a sequence of subsets $\mathbf{H}_n \subset \mathbf{Z}^2$ decreasing to an empty intersection, such that each $\mathbf{H}_n$ is $\mathbf{G}_n$-periodic and the array Ω restricted to $\mathbf{Z}^2 \setminus \mathbf{H}_n$ is $\mathbf{G}_n$-periodic.

The elements of $\mathbf{H}_n$ will be called n-**holes**.

The remainder of this section contains rather obvious observations, so we will omit a detailed argumentation.

If π is an endomorphism of a Toeplitz flow $(\overline{O}(\omega), S)$ which is bijective on the Toeplitz part of the flow (i.e., π is either an automorphism or it is of the first type) then the array Ω defined by

$$\Omega(i,j) = (\pi^j\omega)(i)$$

is a Toeplitz array. Moreover, if h_n is the displacement by which π shifts the n-holes of ω (in other words π induces the translation of the adding machine by $h = (h_n)$) then the subgroups $\mathbf{G}_n$ $(n \geq 1)$ for Ω can be chosen in such a way that for each n

$$(p_n, 0) \text{ and } (-h_n, 1)$$

are in $\mathbf{G}_n$.

Conversely, each row of a given arbitrary Toeplitz array Ω is a Toeplitz sequence with a period structure (p_n) such that for each n, p_n is the smallest natural number with $(p_n, 0) \in \mathbf{G}_n$. The n-holes of these Toeplitz sequences are obtained from the n-holes of the array and vice versa. Now, if Ω is such that in each $\mathbf{G}_n$ there are vectors of the form $(p_n, 0)$ and $(-h_n, 1)$, then all rows of all arrays from the orbit closure of Ω by the $\mathbf{Z}^2$-action generated by the horizontal and vertical shifts are sequences from the same Toeplitz flow $(\overline{O}(\omega), S)$, where ω is the central row of Ω :

$$\omega(i) = \Omega(i, 0).$$

The vertical shift viewed as a transformation on rows corresponds to the translation of the underlying adding machine by the element $h = (h_n)$. This mapping extends to an endomorphism of $(\overline{O}(\omega), S)$ exactly in the case when it happens to coincide with a certain code.

Noncoalescent Toeplitz flow

THEOREM 1: *There exists a regular Toeplitz sequence ω such that the induced Toeplitz flow $(\overline{O}(\omega), S)$ is noncoalescent; more precisely, it admits an endomorphism $\pi\colon \overline{O}(\omega) \to \overline{O}(\omega)$ of the first type.*

Proof: The proof will take up the remainder of this note. At first we define a finite alphabet Σ, next we assign a code $\Pi\colon \Sigma^3 \to \Sigma$ to generate an endomorphism of the shift system, and finally we construct a Toeplitz flow over Σ with the desired properties.

THE ALPHABET Σ. The alphabet will consist of 13 symbols. For convenience while describing the code we let

$$\Sigma = \{\mathrm{F}, \mathrm{M}\} \times \mathbf{Z}_6 \ \cup \ \{*\},$$

where $\mathbf{Z}_6 = \{0,1,2,3,4,5\}$ is the additive cyclic group. The symbols from the product set will be written as $\mathrm{F}_0,\ \mathrm{F}_1,\ \mathrm{M}_2$, etc.

DESCRIBING THE CODE. The code is given by the following rules presented in the inverse-dominance order (i.e., for a given triple choose the last rule that applies):

The star rule.

$$\frac{x\ y\ z}{*},$$

where $x,\ y$ and z represent arbitrary symbols in Σ.

The walking rules.

$$\frac{\mathrm{F}_\alpha\ x\ \ y}{\mathrm{F}_{\alpha+1}},\quad \frac{x\ \ y\ \mathrm{M}_\beta}{\mathrm{M}_{\beta+1}},$$

where $\alpha, \beta \in \mathbf{Z}_6$.

The mating rules.

$$\frac{\mathrm{F}_0\ \mathrm{M}_0\ x}{\mathrm{F}_5},\quad \frac{x\ \mathrm{F}_0\ \mathrm{M}_0}{\mathrm{M}_4}.$$

Remark: In our Toeplitz sequence ω the symbols F_α will appear exclusively at even positions while the symbols M_β appear only at odd ones. Thus, triples like $[\mathrm{F}_\alpha\ x\ \mathrm{M}_\beta]$ will never occur.

Interpretation: F and M stand for "female" and "male", respectively, while $\mathbf{Z}_6$ is some kind of a vital cycle (for both sexes!). As days goes by, females walk to the right, and males walk to the left, their cycles advancing each day. When a couple meets (female on the left, male on the right) at a day when both their cycles are at zero, then they "mate", which resets the female's and male's cycles to 5 and 4, respectively. Note that the mating does not affect the translocation of the individuals. If they meet in any other situation, they walk on without interaction.

The stars fill up the remaining space.

Example: As usual, we write the image of a sequence below it, so that each triple is coded to the entry under its center. The diagram below includes all types of occurrences that we will observe while iterating the code on ω.

$$\begin{array}{lll}
. & \mathrm{F}_5 \ast \ast \mathrm{M}_5 \ast \ast \mathrm{F}_0 \ast \ast \ast \ast \ast \mathrm{F}_5 \ast \ast & . \\
. & \ast \mathbf{F}_0\mathbf{M}_0 \ast \ast \ast \ast \mathrm{F}_1 \ast \ast \ast \ast \ast \mathbf{F}_0\mathbf{M}_0 & . \\
. & \ast \mathrm{M}_4\mathrm{F}_5 \ast \ast \ast \ast \ast \mathrm{F}_2 \ast \ast \ast \ast \mathrm{M}_4\mathrm{F}_5 & . \\
. & \mathrm{M}_5 \ast \ast \mathrm{F}_0 \ast \ast \ast \ast \ast \mathrm{F}_3 \ast \ast \mathrm{M}_5 \ast \ast & . \\
. & \ast \ast \ast \ast \mathrm{F}_1 \ast \ast \ast \ast \ast \mathrm{F}_4\mathrm{M}_0 \ast \ast \ast & . \\
. & \ast \ast \ast \ast \ast \mathrm{F}_2 \ast \ast \ast \ast \mathrm{M}_1\mathrm{F}_5 \ast \ast \mathrm{M}_2 & . \\
. & \mathrm{F}_2 \ast \ast \ast \ast \ast \mathrm{F}_3 \ast \ast \mathrm{M}_2 \ast \ast \mathrm{F}_0\mathrm{M}_3 \ast & . \\
. & \ast \mathrm{F}_3 \ast \ast \ast \ast \ast F_4M_3 \ast \ast \ast \mathrm{M}_4\mathrm{F}_1 \ast & . \\
. & \ast \ast \mathrm{F}_4 \ast \ast \ast \ast \mathrm{M}_4\mathrm{F}_5 \ast \ast \mathrm{M}_5 \ast \ast \mathrm{F}_2 & . \\
. & \ast \ast \ast \mathrm{F}_5 \ast \ast \mathrm{M}_5 \ast \ast \mathbf{F}_0\mathbf{M}_0 \ast \ast \ast \ast & . \\
. & \ast \ast \ast \ast \mathbf{F}_0\mathbf{M}_0 \ast \ast \ast \mathrm{M}_4\mathrm{F}_5 \ast \ast \ast \ast & . \\
. & \ast \ast \ast \ast \mathrm{M}_4\mathrm{F}_5 \ast \ast \mathrm{M}_5 \ast \ast \mathrm{F}_0 \ast \ast \ast & .
\end{array}$$

The "couples" $[\mathrm{F}_0\ \mathrm{M}_0]$ are displayed in boldface. It is seen from the rules that the distance covered by each female between meeting consecutive mates equals $2(\mathrm{mod}\, 6)$, while for a male this distance is equal to $3(\mathrm{mod}\, 6)$.

Notice a pair $[\mathrm{F}_4\ \mathrm{M}_3]$ in the center of the diagram (in italic) which codes to the pair $[\mathrm{M}_4\ \mathrm{F}_5]$ which looks as if it had just mated, while in fact it has not. We will call this case "flirting".

INDUCTIVE CONSTRUCTION OF A TOEPLITZ ARRAY. We shall now construct a Toeplitz array $\Omega(\mathbf{u})$ over the alphabet Σ, with the following properties: vectors of the form $(p_n, 0)$ and $(-h_n, 1)$ are in $\mathbf{G}_n$ for each n, and the vertical shift coincides with the code π. The construction, as usually, goes by induction. The idea is to add in each step a new "caste" within which every meeting of opposite sexes leads to a mating, while mating between castes is prohibited. On the other hand, flirting will occur between different castes of arbitrarily high levels.

For all $n \geq 0$. Let

$$\mathbf{p}_n = (-3^{(2^n)}, 3^{(2^n)}) \quad \text{and} \quad \mathbf{q}_n = (3^{(2^n)} - 1, 3^{(2^n)} - 1).$$

We have $\mathbf{p}_n = 3^{(2^{n-1})}\mathbf{p}_{n-1}$ and $\mathbf{q}_n = (3^{(2^{n-1})}+1)\mathbf{q}_{n-1}$, thus $\mathbf{G}_n \subset \mathbf{G}_{n-1}$. Further, the vectors $-(3^{(2^n)}-1)\mathbf{p}_n + 3^{(2^n)}\mathbf{q}_n$ and $\mathbf{p}_n - \mathbf{q}_n$ have the form $(p_n, 0)$ and $(-h_n, 1)$, respectively.

The induction starts for $n = 1$.

Step $n = 1$. We have $\mathbf{p}_1 = (-9, 9)$ and $\mathbf{q}_1 = (8, 8)$. Let

$$\mathbf{u}_1 = (0, 0).$$

We place the couple $[\mathrm{F}_0\,\mathrm{M}_0]$ at the position $\mathbf{u}_1$, i.e., so that F_0 occupies the origin. Then we copy the couple all over $\mathbf{G}_1$. Now, applying at first the mating rule and then the walking rule we place the symbols F_5, F_0, $\mathrm{F}_1, \ldots$ and M_4, M_5, $\mathrm{M}_0, \ldots$ along diagonal lines extending downward from the couples. Since the vertical distance to be covered by each female equals $8 = 2 (\bmod 6)$, and for males this distance is $9 = 3 (\bmod 6)$, there is no collision with the code at arriving to the couples below. Note that females occupy exclusively even positions on the plane (i.e., with even sum of coordinates), while males occupy only odd ones. We denote by $\mathbf{A}_1$ the subset of $\mathbf{Z}^2$ that is now filled. For the next steps of the construction we reserve the set

$$\mathbf{H}_1 = \mathbf{A}_1 + (-\mathbf{p}_0 - \mathbf{q}_0),$$

which is obviously $\mathbf{G}_1$-periodic and disjoint from $\mathbf{A}_1$ ($-\mathbf{p}_0$ moves the paths of males to other odd positions and $-\mathbf{q}_0$ moves the paths of females to other even positions). The point

$$\mathbf{u}_2 = \mathbf{u}_1 + (-\mathbf{p}_0 - \mathbf{q}_0)$$

plays the role of origin in $\mathbf{H}_1$.

Filling all of the remainder of $\mathbf{Z}^2$ by stars completes the step. (See diagram opposite; the 1-holes are left blank spaces. Since we imagine the output sequence below the input sequence, the vertical axis of $\mathbf{Z}^2$ is directed downward.)

Note that the paths of the males meet the paths of holes where their cycles are at 3, i.e., at places where mating is not possible, but flirting is. Analogously, for females, the corresponding setting is 4.

Step $n \geq 2$. Suppose we have successfully realized our idea for $n-1$ steps. We will set symbols only at the $(n-1)$-holes. Place the couple $[\mathrm{F}_0\,\mathrm{M}_0]$ at $\mathbf{u}_n$ which has been defined in the previous step as the origin of $\mathbf{H}_{n-1}$. Copy the couple all over $\mathbf{G}_n + \mathbf{u}_n$. Fill the diagonals extending downward from these couples according to the mating and walking rules. Again, the distances for the females and males to cover are $3^{(2^n)} - 1 = 2(\mathrm{mod}\,6)$ and $3^{(2^n)} = 3(\mathrm{mod}\,6)$, respectively. Since, by induction, they do not mate with lower castes, there is no collision with the code at arriving to the couples below.

```
. * *   M3* * * * * * * * * * *F4   * * *M4   * * * * * * * * * *   .
. * *M4   * * * * * * * * * *   F5 * *M5* *   * * * * * * * *   .
. *M5* *   * * * * * * * *   * *F0M0* * * *   * * * * * *   * .
.M0* * * *   * * * * * *   * * *M4F5* * * * *   * * * *   * * .
.F5 * * * * *   * * * *   * * *M5* *F0 * * * * *   * *   * * * .
. *F0 * * * * *   * *   * * *M0* * * *F1 * * * * *   * * *M1.
. * *F1 * * * * *   * * *M1* * * * * *F2 * * * *   * *M2* .
. * * *F2 * * * *   * *M2* * * * * * * *F3 * *   * *   M3* * .
. * * * *F3 * *   * *   M3* * * * * * * * * *F4   * * *M4   * * .
. * * * * *F4   * * *M4   * * * * * * * * * *   F5 * *M5* *   * .
. * * * * *   F5 * *M5* *   * * * * * * * *   * *F0M0* * * *   .
. * * * *   * *F0M0* * * *   * * * * * *   * * *M4F5* * * * * .
. * * *   * * *M4F5* * * * *   * * * *   * * *M5* *F0 * * * * .
. * *   * * *M5* *F0 * * * * *   * *   * * *M0* * * *F1 * * * .
. *   * * *M0* * * *F1 * * * * *   * * *M1* * * * * *F2 * * .
.   * * *M1* * * * * *F2 * * * *   * *M2* * * * * * * *F3 * .
.   * *M2* * * * * * * *F3 * *   * *   M3* * * * * * * * * *F4 .
. *   M3* * * * * * * * * *F4   * * *M4   * * * * * * * * * *   .
. *M4   * * * * * * * * * *   F5 * *M5* *   * * * * * * * *   * .
.M5* *   * * * * * * * *   * *F0M0* * * *   * * * * * *   * * .
. * * * *   * * * * * *   * * *M4F5* * * * *   * * * *   * * * .
. * * * * *   * * * *   * * *M5* *F0 * * * * *   * *   * * *M0.
.F0 * * * * *   * *   * * *M0* * * *F1 * * * * *   * * *M1* .
. *F1 * * * * *   * * *M1* * * * * *F2 * * * *   * *M2* * .
. * *F2 * * * *   * *M2* * * * * * * *F3 * *   * *   M3* * * .
. * * *F3 * *   * *   M3* * * * * * * * * *F4   * * *M4   * * *
```

We denote by $\mathbf{A_n}$ the set filled so far in this step (subset of $\mathbf{H}_{n-1}$). Let

$$\mathbf{H}_n = \mathbf{A}_n + (-\mathbf{p}_{n-1} - \mathbf{q}_{n-1}),$$

and denote by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + (-\mathbf{p}_{n-1} - \mathbf{q}_{n-1})$$

the origin of $\mathbf{H}_n$. Since $\mathbf{H}_{n-1}$ is $\mathbf{G}_{n-1}$-periodic, thus $\mathbf{H}_n \subset \mathbf{H}_{n-1}$. Also, by an argument like in step 1, $\mathbf{H}_n$ is disjoint from $\mathbf{A}_n$. Moreover, the place where a male of the n-th caste meets a path of n-holes differs by $-\mathbf{p}_{n-1}$ from his nearest mating place, so his cycle there is

$$-3^{(2^{n-1})} (\operatorname{mod} 6) = 3.$$

Analogously, for females, the difference is $-\mathbf{q}_{n-1}$, so the corresponding setting of her cycle is

$$-3^{(2^{n-1})} + 1 (\operatorname{mod} 6) = 4.$$

Thus, we have excluded mating between the n-th caste and any higher caste. Filling the difference $\mathbf{H}_n \setminus \mathbf{H}_{n-1}$ by stars completes this step of construction.

In search for flirting. Consider the position

$$\mathbf{u}_{n+1} - 2\mathbf{q}_n + \mathbf{q}_{n-1}.$$

It lies on the path of a female of the $(n+1)$-st caste. Since $|-2\mathbf{q}_n + \mathbf{q}_{n-1}| < |\mathbf{q}_{n+1}|$, she is not mating between this place and $\mathbf{u}_{n+1}$, so her cycle there remains undisturbed. Thus the cycle at the considered position equals

$$-2(3^{(2^n)} - 1) + 3^{(2^{n-1})} - 1 (\operatorname{mod} 6) = 4.$$

On the other hand, the same position can be written as

$$\mathbf{u}_n - \mathbf{p}_{n-1} - \mathbf{q}_{n-1} - 2\mathbf{q}_n + \mathbf{q}_{n-1},$$

so it differs from $\mathbf{u}_n$ by $-\mathbf{p}_{n-1}$ (plus an element of $\mathbf{G}_n$). By an earlier computation this is a place, where a male of the n-th caste meets a female of a higher caste while his cycle is at 3. Thus, we have found that flirting occurs between the n-th and $(n+1)$-st caste.

Final notes. It is seen that the rectangle around $(0,0)$ completely filled in each step expands in all directions, so a Toeplitz array has been defined. By the remarks following the definition of a Toeplitz array, the upward shift of our array corresponds to an endomorphism π of the Toeplitz flow $(\overline{O}(\omega), S)$, where ω is the central row of Ω.

It is also obvious that the density of n-holes on the plane decreases to zero, and hence ω is a regular Toeplitz sequence. The structure of n-holes, however, is rich enough, as required in Lemma 1 of [D-K-L]; for any n we can find n-intervals of any even length between zero and h_n.

NONCOALESCENCE OF $(\overline{O}(\omega), S)$. This time the idea is to notice that in each nontoeplitz array Ω' from the orbit closure of Ω by the $\mathbf{Z}^2$-action, there appears an aperiodic part which consists of either a path of a single female (called the "queen") or a path of a single male (the "king") or both. The queen may mate only with the king and vice versa; however, they need not mate at all. The royal pair may also flirt or totally ignore each other, depending on the choice of Ω'. (All the above follows, by standard approximation arguments, from the construction of Ω.) In each row of such an array, i.e., in each nontoeplitz sequence in $(\overline{O}(\omega), S)$, we then see either the queen or the king or both of them in a given day, without seeing their past (the future is always determined by the present). The noninvertibility of the endomorphism becomes manifest for sequences where the royal pair is seen in the "parting position" $[\mathrm{M}_4\ \mathrm{F}_5]$. There are two possibilities: they have either mated or flirted. Observe that for the n-th caste all places where the individuals of higher castes meet "look the same", i.e., they have the same position modulo $\mathbf{G}_n$, hence *the past of the royal pair cannot be deduced* from their position with respect to the castes. Thus, two preimages are possible for the same sequence.

More precisely, let ω' and ω'' be the central rows of

$$\Omega' = \lim_n S^{\mathbf{u}_{n+1}}\Omega,$$
$$\Omega'' = \lim_n S^{\mathbf{u}_{n+1}-2\mathbf{q}_n+\mathbf{q}_{n-1}}\Omega,$$

respectively, where $S^{\mathbf{v}}$ shifts the array Ω so that $\Omega(\mathbf{v})$ is moved to the origin. Then

$$[\omega'(0)\ \omega'(1)] = [\mathrm{F}_0\ \mathrm{M}_0],$$
$$[\omega''(0)\ \omega''(1)] = [\mathrm{F}_4\ \mathrm{M}_3],$$

and the sequences are equal everywhere else. Thus they code to the same image. So, checking the list of the desired properties of the flow $(\overline{O}(\omega), S)$ is now completed. ∎

Fibers over the maximal uniformly continuous factor

Below we sketch the structure of the flow $(\overline{O}(\omega), S)$, constructed above, with respect to the maximal uniformly continuous factor. The details of the deductions are of the same type as presented in previous sections and they will be omitted. Firstly we omit checking the elementary fact that the group G_p is not just any uniformly continuous factor but the maximal one.

If an n-hole is specified modulo $\mathbf{G}_n$ then we can

(1) determine the sex of who will occupy this place (by checking evenness of this position);

(2) for even positions, we can determine the cycle of the female modulo 2, and for odd positions, we can determine the male's cycle modulo 3 (use the fact that the vertical lengths of the generators $\mathbf{p}_t$ and $\mathbf{q}_t$ are 2 and 3 $(\bmod 6)$, respectively).

Thus, if a single hole of infinite level occurs in $\omega' \in \overline{O}(\omega)$, then it can be filled either in two or in three different ways. This indicates the existence of two-element and three-element fibers over the maximal uniformly continuous factor. If there are two holes of infinite level in some ω', then one of them is prescribed for the queen and the other one for the king. The distance between them restricts the number of combinations of their cycles to 5, if the queen is on the left, and to 4 if she is on the right. This indicates the existence of four-element and five-element fibers. Of course, the Toeplitz sequences (with no holes of infinite level) constitute one-element fibers. This exhausts the variety. The endomorphism π sends all fibers to fibers of the same cardinality except that some of the five-element fibers are transformed to four-element fibers. (Mating and flirting of the royal pair occur in the five-element fibers. The other three combinations there are: [F_0 M_3], [F_2 M_3], and [F_4 M_0].)

It still remains unsolved whether there exists noncoalescence of the second type for Toeplitz flows, i.e., whether some of the one-element fibers may admit more rich preimages. If not, we would have obtained that all endomorphisms are invertible on the Toeplitz part of the flow. This conjecture is currently puzzling the author. So far, all attempts to build an appropriate counterexample have failed.

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References

[B-K1] W. Bułatek and J. Kwiatkowski, *The topological centralizers of Toeplitz flows and their $\mathbf{Z}_2$-extensions*, Publications Mathématiques de l'IHES **34** (1990), 45–65.

[B-K2] W. Bułatek and J. Kwiatkowski, *Strictly ergodic Toeplitz flows with positive entropies and trivial centralizers*, Studia Mathematica **103** (1992), 133–142.

[D1] T. Downarowicz, *How a function on a zero-dimensional group $\Delta_{\underline{a}}$ defines a Toeplitz flow*, Bulletin of the Polish Academy of Sciences **38** (1990), 219–222.

[D2] T. Downarowicz, *The Choquet simplex of invariant measures for minimal flows*, Israel Journal of Mathematics **74** (1991), 241–256.

[D-K-L] T. Downarowicz, J. Kwiatkowski and Y. Lacroix, *A criterion for Toeplitz flows to be topologically isomorphic and applications*, Colloquium Mathematicum **68** (1995), 219–228.

[D-L] T. Downarowicz and Y. Lacroix, *A nonregular Toeplitz flow with preset pure point spectrum*, Studia Mathematica, to appear.

[I] A. Iwanik, *Toeplitz flows with pure point spectrum*, Studia Mathematica **118** (1996), 27–35.

[I-L] A. Iwanik and Y. Lacroix, *Some constructions of strictly ergodic non-regular Toeplitz flows*, Studia Mathematica **110** (1994), 191–203.

[J-K] K. Jacobs and M. Keane, *0-1 sequences of Toeplitz type*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **13** (1969), 123–131.

[O] J. C. Oxtoby, *Ergodic sets*, Bulletin of the American Mathematical Society **58** (1952), 116–136.

[W] S. Williams, *Toeplitz minimal flows which are not uniquely ergodic*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **67** (1984), 95–107.